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# The Gaussian Random Walk on the Heisenberg Group.

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THE GAUSSIAN RANDOM WALK ON THE HEISENBERG  
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The Gaussian Random Walk on the Heisenberg Group

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in

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by

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## Abstract

In this dissertation the transience of the random walk associated with the Gaussian probability density function on the Heisenberg group is analyzed and compared with that of a corresponding random walk on the Lie algebra of the Heisenberg group. Asymptotic results are obtained that yield a quantitative reflection of the effect of the non-commutative nature of the group multiplication law on the behavior of the Gaussian random walk.

## Introduction

Let  $G$  be a locally compact group and let  $\mu$  be a probability measure on  $G$ .  $\mu$  generates a random walk on  $G$  with the crucial property that the number of times that the walk can be expected to visit a Borel subset  $A$  of  $G$  is  $\sum \mu^{*m}(A)$  where the summation runs over all positive integers  $m$ , and  $\mu^{*m}$  denotes the  $m$ -fold convolution power of  $\mu$ . It is known [1] that either  $\sum \mu^{*m}(A)$  is finite for every compact set  $A$  or else is infinite for every open set  $A$ . In the former case the random walk determined by  $\mu$  is said to be transient and to "wander to  $\infty$ ." In the latter case the walk is said to be recurrent.

For convenience we now restrict our discussion to the case in which  $G$  is a 1-connected nilpotent Lie group. Such groups are diffeomorphic to  $\mathbb{R}^n$  for some  $n$ . If the group actually is  $\mathbb{R}^n$  then it is a classical theorem (see e.g. [2]) that for  $n \geq 3$  every random walk on  $G$  generated by a measure that is supported by no proper closed subgroup of  $G$  is transient. A special case of a theorem proved by Guivarc'h and Keane [3] is that this result is also true for 1-connected nilpotent Lie groups of dimension greater than two.

Let  $G$  be a 1-connected nilpotent Lie group, let  $L$  denote its Lie algebra, and let  $\exp$  denote the exponential map from  $L$  to  $G$ .  $\exp$  is a global diffeomorphism that carries Euclidean measure on  $L$  to Haar measure on  $G$  [4]. Thus  $\exp$

carries absolutely continuous probability measures on  $G$  to absolutely continuous probability measures on  $L$ . If  $\mu$  is a probability distribution on  $G$  and if  $A$  is a neighborhood of the origin in  $L$  consider the two infinite series  $\sum \int_A \phi^{*m} \circ \exp(l) dl$  and  $\sum \int_A \phi \circ \exp^{*m}(l) dl$ , the former series being associated with the expected number of visits of a random walk on  $G$  to a neighborhood of the origin and the latter series being similarly associated with a random walk on the abelian group structure of  $L$ . Both series are convergent, but one might expect that, since in order to return to the origin in the non-abelian case one must retrace one's steps in the correct order, the first series should converge more rapidly than the second. In this paper we address this question in a particular case--that in which  $\phi \circ \exp$  is the Gaussian distribution and  $G$  is the Heisenberg group.

Our main results are theorems 4.1 and 4.5. The former theorem shows that, as in the abelian case, the behavior of the series in question depends only on the behavior of the convolution powers of the density function evaluated at the origin. The latter theorem shows that, asymptotically, the ratio of  $\phi^{*m} \circ \exp(0,0,0)$  to  $\phi \circ \exp^{*m}(0,0,0)$  behaves like  $m^{-1/2}$ . Roughly, this means that the ratio of the probability of returning to the origin at the  $m^{\text{th}}$  step on the Heisenberg group to the probability of returning to the origin at the  $m^{\text{th}}$  step on  $R^3$  behaves like  $m^{-1/2}$ . We thus have a quantitative reflection of the effect of the non-commutative group



multiplication on the difficulty of returning to the origin.

One of the main tools in the analysis of random walks on  $R^n$  is the Fourier transform. Its value in this situation is that it carries convolution to pointwise multiplication. In the interesting but relatively simple case in which the distribution function  $\phi$  is a Schwartz function one can then apply the Plancherel theorem (inversion theorem) to analyze the series  $\sum_A \phi^{*m}(x) dx$ . One might thus expect that in the case in which  $G$  is a 1-connected nilpotent Lie group that the representation-theoretic Fourier transform and inversion theorem would prove to be similarly useful. This has not, thus far, been the case. In the non-abelian setting the inversion theorem takes the form, for  $\phi$  a Schwartz function, [5]

$$\phi(1) = \int_{\hat{G}} \hat{\phi}(\pi) d\pi$$

where  $\hat{G}$  is the set of unitary-equivalence classes of irreducible unitary representations of  $G$ ,  $d\pi$  is a (known) positive measure and

$$\hat{\phi}(\pi) = \text{trace } \pi_\phi = \text{trace } \int_G \pi(g)\phi(g) dg.$$

While the map  $\phi \mapsto \pi_\phi$  is multiplicative, the trace function is not, unless  $\pi$  is one-dimensional. Therefore the non-abelian Fourier transform is not multiplicative.

Since we have been unable to utilize the non-abelian Fourier transform, our analysis of the problem at hand

proceeds by real-variable methods. The proofs are computational, using classical theorems. One property of the Gaussian density that makes our calculations manageable is that it is its own (abelian) Fourier transform. This fact is crucial to the proof of lemma 2.5.

## Chapter I

### Preliminaries

In this section we list for convenience certain notations and conventions that will be used in the sequel, indicate the means by which convolutions on the Lie group can be lifted to the Lie algebra, and consider briefly the abelian version of our random walk.

$N_3$  will denote the Heisenberg group. This group can be thought of as the group of  $3 \times 3$  unipotent matrices, or alternatively as  $R^3$  with multiplication given by

$$(x,y,z)(x',y',z') = (x+x',y+y',z+z'+xy').$$

The Lie algebra of  $N_3$  is  $R^3$  with bracket operation given by

$$[(x,y,z), (x',y',z')] = (0,0,xy'-yx').$$

If  $G$  is any 1-connected nilpotent Lie group with Lie algebra  $L$  then  $\exp$  is a global diffeomorphism and there is a map  $C-H:L \rightarrow G$ , called the Campbell-Hausdorff map [4] such that for  $l, l' \in L$

$$C-H(l, l') = \exp^{-1}(\exp l \exp l') = l + l' + \frac{1}{2}[l, l'] + \dots$$

where  $\dots$  denotes higher order bracket terms. Since the group being considered is nilpotent the series terminates, and for  $N_3$  all of the higher order bracket terms vanish.

Recall that  $\exp$  carries Euclidean measure  $dl$  on the Lie algebra to Haar measure  $dg$  on  $G$ . Let  $f \in L^1(G)$  and let  $l_i = \exp^{-1}g_i$ . Then

$$\begin{aligned}
f^{*m}(g_0) &= \int_G f(g_0 g_1^{-1}) f^{*m-1}(g_1) dg_1 \\
&\vdots \\
&= \int_G \cdots \int_G f(g_0 g_1^{-1}) \cdots f(g_{m-2} g_{m-1}^{-1}) f(g_{m-1}) dg_1 \cdots dg_{m-1} \\
&= \int_L \cdots \int_L f \circ \exp(C-H(l_0, -l_1)) \cdots f \circ \exp(C-H(l_{m-2}, -l_{m-1})) \\
&\quad f \circ \exp(l_{m-1}) dl_1 \cdots dl_{m-1}
\end{aligned}$$

Thus if one knows C-H explicitly one can carry out all of the integrations necessary to compute convolutions on the group by lifting to the algebra.

We note that for the Lie algebra of  $N_3$

$$C-H((x,y,z),(x',y',z')) = (x+x', y+y', z+z'+\frac{1}{2}(xy'-yx')) .$$

For notational convenience we introduce here certain conventions that will remain in force throughout the remainder of the paper:

1. All measures on Euclidean space are taken to be Lebesgue measure normalized by multiplication by  $(2\pi)^{-\frac{1}{2}n}$  where  $n$  is the dimension of the space in question.
2. For any  $n$ ,  $E(x_1, \dots, x_n) = e^{-i(x_1 + \dots + x_n)}$ .
3. For any  $n$ ,  $G(x_1, \dots, x_n) = e^{-\frac{1}{2}(x_1^2 + \dots + x_n^2)}$ .
4. For any  $n$ ,  $F(x_1, \dots, x_n) = e^{-\frac{1}{2}(x_1 + \dots + x_n)}$ .
5. Sums taken over an empty indexing set are taken to be 0.
6. Products taken over an empty indexing set are taken to be 1.

6.  $\phi:N_3 \rightarrow R$  is the function defined by

$$\phi \circ \exp(x,y,z) = G(x,y,z).$$

We call the random walk on  $N_3$  associated with the measure  $\phi dg$  the Gaussian random walk on  $N_3$ .

We conclude this section by considering briefly the abelian Gaussian random walk, that is, the random walk on  $R^3$  associated with the measure  $G(x,y,z)dx dy dz$ . It is easily computed, either by direct computation or by utilizing the fact that  $\hat{G}=G$ , that

$$G^{*m}(x,y,z) = m^{-3/2} G(m^{-1/2}x, m^{-1/2}y, m^{-1/2}z).$$

It follows that, denoting the sphere centered at the origin of radius  $k>0$  by  $B_k$ , for every  $m$  and for all  $(x,y,z) \in B_k$

$$G^{*m}(0,0,0) \geq G^{*m}(x,y,z) \geq G(k)G^{*m}(0,0,0).$$

Thus for fixed  $k>0$ ,

$$\int_{B_k} G^{*m}(x,y,z) dx dy dz = O(G^{*m}(0,0,0))$$

and

$$G^{*m}(0,0,0) = O\left(\int_{B_k} G^{*m}(x,y,z) dx dy dz\right).$$

The question of the transience and rate of "wandering to  $\infty$ " of this random walk are thus seen to depend only on the behavior of  $G^{*m}(0,0,0)$  as  $m \rightarrow \infty$ .

## Chapter II

### Evaluation of $\phi^{*m} \cdot \exp^{\wedge}$

Our goal in this section is the evaluation of  $\phi^{*m} \cdot \exp^{\wedge}$ . To do this we define a particular sequence of polynomials, prove some preparatory results concerning their algebraic interrelationships, and then proceed with the evaluation of  $\phi^{*m} \cdot \exp^{\wedge}$  by means of a rather long and involved calculation. The major results of this section are proposition 2.2 and theorem 2.6 and its corollaries.

DEFINITION 2.1 For  $m=0,1,\dots$  define  $P_m \in R(\gamma)$  recursively by

$$P_0(\gamma) = P_1(\gamma) = 1$$

and for  $m \geq 1$

$$\begin{aligned} 2.1 \quad P_{m+1}(\gamma)/P_m(\gamma) &= P_m(\gamma)/P_{m-1}(\gamma) \\ &\quad - \gamma^4(16P_{m-1}(\gamma)P_m(\gamma))^{-1} \left( \sum_{i=0}^{m-2} P_i(\gamma) \right)^2 \\ &\quad + \gamma^2 P_{m-1}(\gamma)/(4P_m(\gamma)) . \end{aligned}$$

It is not clear a priori that the above definition actually makes sense since the possibility that  $P_m=0$  for some  $m$  has not been ruled out. In the next proposition we show that not only does the above definition make sense, but

also that  $P_m$  is a polynomial with  $P_m(\gamma) \geq 1$  for all real  $\gamma$ .

PROPOSITION 2.2 For  $m=0,1,\dots$   $P_{m+1} \in R[\gamma]$ ,  $P_{m+1}(0)=1$ , and

$$2.2 \quad P_{m+1}(\gamma) = P_m(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-1} P_i(\gamma)$$

PROOF

We proceed by induction on  $m$ , the case  $m=0$  being trivial. Assuming the proposition to be true for non-negative integers  $\leq m$ , definition 2.1 makes sense and we then have by 2.1

$$P_{m+2}(\gamma) = (P_m(\gamma))^{-1} (P_{m+1}^2(\gamma) - \frac{\gamma^4}{16} (\sum_{i=0}^{m-1} P_i(\gamma))^2 + \frac{\gamma^2}{4} P_m^2(\gamma))$$

Applying the inductive hypothesis we see that

$$\begin{aligned} P_{m+2}(\gamma) &= (P_m(\gamma))^{-1} \left[ (P_m(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-1} P_i(\gamma))^2 \right. \\ &\quad \left. - \frac{\gamma^4}{16} (\sum_{i=0}^{m-1} P_i(\gamma))^2 + \frac{\gamma^2}{4} P_m^2(\gamma) \right] \\ &= (P_m(\gamma))^{-1} \left[ (1 + \frac{\gamma^2}{4}) P_m^2(\gamma) + \frac{\gamma^2}{2} P_m(\gamma) \sum_{i=0}^{m-1} P_i(\gamma) \right] \\ &= P_m(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-1} P_i(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^m P_i(\gamma) \\ &= P_{m+1}(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^m P_i(\gamma) \end{aligned}$$

Thus 2.2 holds with  $m+1$  in place of  $m$ . We conclude from 2.2 and the inductive hypothesis that  $P_{m+2} \in R[\gamma]$  and that

$$P_{m+2}(0) = P_{m+1}(0) = 1.$$

This completes the proof of the proposition.

The following corollary follows trivially from definition 2.1 (for the case  $m=0$ ) and from proposition 2.2 and induction (for  $m \geq 1$ ).

COROLLARY 2.3 For  $m=0,1,\dots$   $P_m \in R[\gamma^2/4]$  with non-negative coefficients and constant coefficient 1. In particular, for every  $m$  and every real  $\gamma$ ,

$$2.3 \quad P_m(\gamma) \geq 1.$$

The next lemma will be used in the proof of lemma 2.5.

LEMMA 2.4 For  $m=0,1,\dots$

$$2.4 \quad (P_m(\gamma))^{-1} \sum_{i=0}^{m-1} P_i(\gamma) - \gamma^2 (4P_m(\gamma)P_{m+1}(\gamma))^{-1} \left( \sum_{i=0}^{m-1} P_i(\gamma) \right)^2 \\ = (P_{m+1}(\gamma))^{-1} \sum_{i=0}^{m-1} P_i(\gamma)$$

PROOF

We apply lemma 2.2 and calculate.

$$(P_m(\gamma))^{-1} \sum_{i=0}^{m-1} P_i(\gamma) - \gamma^2 (4P_m(\gamma)P_{m+1}(\gamma))^{-1} \left( \sum_{i=0}^{m-1} P_i(\gamma) \right)^2 \\ = (P_m(\gamma)P_{m+1}(\gamma))^{-1} \left[ P_{m+1}(\gamma) \sum_{i=0}^{m-1} P_i(\gamma) - \frac{\gamma^2}{4} \left( \sum_{i=0}^{m-1} P_i(\gamma) \right)^2 \right] \\ = (P_m(\gamma)P_{m+1}(\gamma))^{-1} \left[ P_m(\gamma) \sum_{i=0}^{m-1} P_i(\gamma) + \frac{\gamma^2}{4} \left( \sum_{i=0}^{m-1} P_i(\gamma) \right)^2 \right. \\ \left. - \frac{\gamma^2}{4} \left( \sum_{i=0}^{m-1} P_i(\gamma) \right)^2 \right] \\ = (P_m(\gamma))^{-1} \sum_{i=0}^{m-1} P_i(\gamma)$$



LEMMA 2.5 Fix  $m \in \{1, 2, \dots\}$ . Let  $k$  be an integer with  $0 \leq k \leq m$ .

Then

$$\begin{aligned}
 2.5 \quad \phi^{*m} \circ \exp^{\wedge}(\alpha, \beta, \gamma) &= \int_{\mathbb{R}^{2(m-k)}} E \left[ \sum_{j=0}^{m-k-1} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l \right) x_j, \right. \\
 &\quad \left. \sum_{j=0}^{m-k-1} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l \right) y_j \right] \\
 &\quad \prod_{j=0}^{m-k-1} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} x_j^2 \right. \\
 &\quad \left. + (P_k(\gamma))^{-1} (\beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l) \right. \\
 &\quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) x_j \right] \\
 &\quad \prod_{j=0}^{m-k-1} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} y_j^2 \right. \\
 &\quad \left. - (P_k(\gamma))^{-1} (\alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l) \right. \\
 &\quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) y_j \right] \\
 &\quad F \left[ (P_k(\gamma))^{-1} \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) (\alpha^2 + \beta^2) \right. \\
 &\quad \left. + m \gamma^2 \right] \\
 &\quad (P_k(\gamma))^{-1} \prod_{i=0}^{m-k-1} dx_i dy_i
 \end{aligned}$$

PROOF

We proceed by induction on  $k$ . Consider first the case  $k=0$ .

$$\phi^{*m} \circ \exp^{\wedge}(\alpha, \beta, \gamma) = \int_{\mathbb{R}^3} E(\alpha x_0, \beta y_0, \gamma z_0) \phi^{*m} \circ \exp(x_0, y_0, z_0) dx_0 dy_0 dz_0$$

$$= \int_{R^{3m}} E(\alpha x_0, \beta y_0, \gamma z_0) \prod_{i=0}^{m-2} G(x_i - x_{i+1}, y_i - y_{i+1}, z_i - z_{i+1} + \frac{1}{2}(x_{i+1}y_i - x_i y_{i+1})) \\ G(x_{m-1}, y_{m-1}, z_{m-1}) \prod_{i=0}^{m-1} dx_i dy_i dz_i .$$

Making several changes of variable the above integral becomes

$$\int_{R^{3m}} E\left[\alpha x_0, \beta y_0, \gamma\left(z_0 + \frac{1}{2} \sum_{i=0}^{m-2} (x_i y_{i+1} - x_{i+1} y_i)\right)\right] \\ \prod_{i=0}^{m-2} G(x_i - x_{i+1}, y_i - y_{i+1}, z_i - z_{i+1}) G(x_{m-1}, y_{m-1}, z_{m-1}) \\ \prod_{i=0}^{m-1} dx_i dy_i dz_i .$$

Integrating with respect to  $dz_0 \dots dz_{m-1}$  the integral becomes, noting that  $\hat{G}=G$  and  $G(x, y, z)=G(x)G(y)G(z)$ ,

$$\int_{R^{2m}} E\left[\alpha x_0, \beta y_0, \frac{\gamma}{2} \sum_{i=0}^{m-2} (x_i y_{i+1} - x_{i+1} y_i)\right] G(x_{m-1}) \\ \prod_{i=0}^{m-2} G(x_i - x_{i+1}) G(y_{m-1}) \prod_{i=0}^{m-2} G(y_i - y_{i+1}) G(\sqrt{m} \gamma) \\ \prod_{i=0}^{m-1} dx_i dy_i .$$

Again making several changes of variable we rewrite the integral as

$$\int_{R^{2m}} E\left[\sum_{j=0}^{m-1} \left(\alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l\right) x_j, \sum_{j=0}^{m-1} \left(\beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l\right) y_j\right] \\ \prod_{j=0}^{m-1} F(x_j^2) \prod_{j=0}^{m-1} F(y_j^2) F(m\gamma^2) \prod_{i=0}^{m-1} dx_i dy_i .$$

This proves the lemma in the case  $k=0$ .

Next, for  $0 \leq k < m$ ,

$$\begin{aligned}
 & \int_{R^{2(m-k)}} E \left[ \sum_{j=0}^{m-k-1} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l \right) x_j, \sum_{j=0}^{m-k-1} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l \right) y_j \right] \\
 & \sum_{j=0}^{m-k-1} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} x_j^2 + (P_k(\gamma))^{-1} \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l \right) \right. \\
 & \quad \left. \cdot \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) x_j \right] \\
 & \sum_{j=0}^{m-k-1} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} y_j^2 - (P_k(\gamma))^{-1} \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l \right) \right. \\
 & \quad \left. \cdot \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) y_j \right] \\
 & F \left[ (P_k(\gamma))^{-1} \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) (\alpha^2 + \beta^2) + m\gamma^2 \right] (P_k(\gamma))^{-1} \\
 & \quad \sum_{i=0}^{m-k-1} dx_i dy_i
 \end{aligned}$$

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$$\begin{aligned}
&= \int_{R^{2(m-k)}} E \left[ \sum_{j=0}^{m-k-2} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l \right) x_j, \sum_{j=0}^{m-k-2} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l \right) y_j \right] \\
&\quad \prod_{j=0}^{m-k-2} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} x_j^2 + (P_k(\gamma))^{-1} \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l \right) \right. \\
&\quad \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) x_j \right] \\
&\quad \prod_{j=0}^{m-k-2} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} y_j^2 - (P_k(\gamma))^{-1} \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l \right) \right. \\
&\quad \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) y_j \right] \\
&\quad F \left[ (P_k(\gamma))^{-1} \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) (\alpha^2 + \beta^2) + m\gamma^2 \right] \\
&\quad E \left[ \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{m-k-2} y_l \right) x_{m-k-1}, \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{m-k-2} x_l \right) y_{m-k-1} \right] \\
&\quad F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} x_{m-k-1}^2 + (P_k(\gamma))^{-1} \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} x_l \right) \right. \\
&\quad \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) x_{m-k-1} \right] \\
&\quad F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} y_{m-k-1}^2 - (P_k(\gamma))^{-1} \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} y_l \right) \right. \\
&\quad \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) y_{m-k-1} \right] \\
&\quad (P_k(\gamma))^{-1} \prod_{i=0}^{m-k-1} dx_i dy_i.
\end{aligned}$$

Completing the square in the arguments of the last two F's in the integrand above the integral becomes

$$\begin{aligned}
& \int_{R^{2(m-k)}} E \left[ \sum_{j=0}^{m-k-2} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l \right) x_j, \sum_{j=0}^{m-k-2} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l \right) y_j \right] \\
& \prod_{j=0}^{m-k-2} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} x_j^2 + (P_k(\gamma))^{-1} \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l \right) \right. \\
& \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) x_j \right] \\
& \prod_{j=0}^{m-k-2} F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} y_j^2 - (P_k(\gamma))^{-1} \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l \right) \right. \\
& \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) y_j \right] \\
& F \left[ (P_k(\gamma))^{-1} \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) (\alpha^2 + \beta^2) + m\gamma^2 \right] \\
& E \left[ \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{m-k-2} y_l \right) x_{m-k-1}, \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{m-k-2} x_l \right) y_{m-k-1} \right] \\
& F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} \left\{ x_{m-k-1} + (2P_{k+1}(\gamma))^{-1} \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} x_l \right) \right. \right. \\
& \quad \left. \left. \sum_{l=0}^{k-1} P_l(\gamma) \right\}^2 \right] \\
& F \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} \left\{ y_{m-k-1} - (2P_{k+1}(\gamma))^{-1} \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} y_l \right) \right. \right. \\
& \quad \left. \left. \sum_{l=0}^{k-1} P_l(\gamma) \right\}^2 \right] \\
& F \left[ (4P_k(\gamma)P_{k+1}(\gamma))^{-1} \left( \sum_{l=0}^{k-1} P_l(\gamma) \right)^2 \left\{ -\beta^2 \gamma^2 - \beta \gamma^3 \sum_{l=0}^{m-k-2} x_l \right. \right. \\
& \quad \left. \left. - \frac{\gamma^4}{4} \left( \sum_{l=0}^{m-k-2} x_l \right)^2 - \alpha^2 \gamma^2 + \alpha \gamma^3 \sum_{l=0}^{m-k-2} y_l - \frac{\gamma^4}{4} \left( \sum_{l=0}^{m-k-2} y_l \right)^2 \right\} \right] \\
& (P_k(\gamma))^{-1} \prod_{i=0}^{m-k-1} dx dy .
\end{aligned}$$

Translating  $x_{m-k-1}$  and  $y_{m-k-1}$  appropriately and performing some algebraic manipulations we rewrite the integral as

$$\begin{aligned}
& \int_{R^{2(m-k)}} E \left[ \sum_{j=0}^{m-k-2} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l \right) x_j, \sum_{j=0}^{m-k-2} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l \right) y_j \right] \\
& \prod_{j=0}^{m-k-2} F \left[ \{ P_{k+1}(\gamma) (P_k(\gamma))^{-1} - \gamma^4 (16 P_k(\gamma) P_{k+1}(\gamma))^{-1} \right. \\
& \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right)^2 \} x_j^2 \right. \\
& \quad + \{ (P_k(\gamma))^{-1} (\beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l) \sum_{l=0}^{k-1} P_l(\gamma) \\
& \quad - \gamma^2 (4 P_k(\gamma) P_{k+1}(\gamma))^{-1} (\beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l) \left( \sum_{l=0}^{k-1} P_l(\gamma) \right)^2 \} x_j \Big] \\
& \prod_{j=0}^{m-k-2} F \left[ \{ P_{k+1}(\gamma) (P_k(\gamma))^{-1} - \gamma^4 (16 P_k(\gamma) P_{k+1}(\gamma))^{-1} \right. \\
& \quad \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right)^2 \} y_j^2 \right. \\
& \quad - \{ (P_k(\gamma))^{-1} (\alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l) \sum_{l=0}^{k-1} P_l(\gamma) \\
& \quad - \gamma^2 (4 P_k(\gamma) P_{k+1}(\gamma))^{-1} (\alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l) \left( \sum_{l=0}^{k-1} P_l(\gamma) \right)^2 \} y_j \Big] \\
& F \left[ \{ P_k(\gamma) \}^{-1} \sum_{l=0}^{k-1} P_l(\gamma) - \gamma^2 (4 P_k(\gamma) P_{k+1}(\gamma))^{-1} \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) \right. \\
& \quad \left. (\alpha^2 + \beta^2) + m \gamma^2 \right] \\
& E \left[ \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{m-k-2} y_l \right) \{ x_{m-k-1} - (2 P_{k+1}(\alpha))^{-1} \right. \\
& \quad \left. (\beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} x_l) \sum_{l=0}^{k-1} P_l(\gamma) \} \right] \\
& E \left[ \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{m-k-2} x_l \right) \{ y_{m-k-1} + (2 P_{k+1}(\gamma))^{-1} \right. \\
& \quad \left. (\alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} y_l) \sum_{l=0}^{k-1} P_l(\gamma) \} \right] \\
& G \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} x_{m-k-1}^2, P_{k+1}(\gamma) (P_k(\gamma))^{-1} y_{m-k-1}^2 \right] \\
& (P_k(\gamma))^{-1} \prod_{i=0}^{m-k-1} dx_i dy_i.
\end{aligned}$$

Noting that

$$\begin{aligned} & \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{m-k-2} x_l \right) \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} y_l \right) \\ & - \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{m-k-2} y_l \right) \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{m-k-2} x_l \right) = 0 \end{aligned}$$

and making the substitutions

$$u = \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} \right]^{\frac{1}{2}} x_{m-k-1}, \quad v = \left[ P_{k+1}(\gamma) (P_k(\gamma))^{-1} \right]^{\frac{1}{2}} y_{m-k-1}$$

the above integral becomes upon applying lemma 2.4 and then integrating with respect to  $du dv$ ,

$$\int_{R^{2(m-k-1)}} E \left[ \sum_{j=0}^{m-k-2} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l \right) x_j, \sum_{j=0}^{m-k-2} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l \right) y_j \right]$$

$$\prod_{j=0}^{m-k-2} F \left[ \{ P_{k+1}(\gamma) (P_k(\gamma))^{-1} - \gamma^4 (16 P_k(\gamma) P_{k+1}(\gamma))^{-1} \right. \\ \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right)^2 \} x_j^2 \right]$$

$$+ \{ (P_{k+1}(\gamma))^{-1} \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l \right) \sum_{l=0}^{k-1} P_l(\gamma) \} x_j \}$$

$$\prod_{j=0}^{m-k-2} F \left[ \{ P_{k+1}(\gamma) (P_k(\gamma))^{-1} - \gamma^4 (16 P_k(\gamma) P_{k+1}(\gamma))^{-1} \right. \\ \left. \left( \sum_{l=0}^{k-1} P_l(\gamma) \right)^2 \} y_j^2 \right]$$

$$- \{ (P_{k+1}(\gamma))^{-1} \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l \right) \sum_{l=0}^{k-1} P_l(\gamma) \} y_j \}$$

$$F \left[ (P_{k+1}(\gamma))^{-1} \left( \sum_{l=0}^{k-1} P_l(\gamma) \right) (\alpha^2 + \beta^2) + m \gamma^2 \right]$$

$$F \left[ P_k(\gamma) (P_{k+1}(\gamma))^{-1} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{m-k-2} x_l \right)^2 \right]$$

$$F \left[ P_k(\gamma) (P_{k+1}(\gamma))^{-1} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{m-k-2} y_l \right)^2 \right]$$

$$(P_{k+1}(\gamma))^{-1}$$

$$\prod_{i=0}^{m-k-2} dx_i dy_i .$$

Performing some algebraic manipulations and applying 2.1 the integral becomes, finally

$$\begin{aligned}
 & \int_{R^{2(m-k-1)}} E \left[ \sum_{j=0}^{m-k-2} \left( \alpha - \frac{\gamma}{2} \sum_{l=0}^{j-1} y_l \right) x_j, \sum_{j=0}^{m-k-2} \left( \beta + \frac{\gamma}{2} \sum_{l=0}^{j-1} x_l \right) y_j \right] \\
 & \prod_{j=0}^{m-k-2} F \left[ P_{k+2}(\gamma) (P_{k+1}(\gamma))^{-1} x_j^2 + (P_{k+1}(\gamma))^{-1} \left( \beta \gamma + \frac{\gamma^2}{2} \sum_{l=0}^{j-1} x_l \right) \right. \\
 & \qquad \qquad \qquad \left. \left( \sum_{l=0}^k P_l(\gamma) \right) x_j \right] \\
 & \prod_{j=0}^{m-k-2} F \left[ P_{k+2}(\gamma) (P_{k+1}(\gamma))^{-1} y_j^2 - (P_{k+1}(\gamma))^{-1} \left( \alpha \gamma - \frac{\gamma^2}{2} \sum_{l=0}^{j-1} y_l \right) \right. \\
 & \qquad \qquad \qquad \left. \left( \sum_{l=0}^k P_l(\gamma) \right) y_j \right] \\
 & F \left[ (P_{k+1}(\gamma))^{-1} \left( \sum_{l=0}^k P_l(\gamma) \right) (\alpha^2 + \beta^2) + m\gamma^2 \right] \\
 & (P_{k+1}(\gamma))^{-1} \prod_{i=0}^{m-k-2} dx_i dy_i .
 \end{aligned}$$

This provides the inductive step and completes the proof of the lemma.

The case  $k=m$  of lemma 2.5 is important. We state this special case as a theorem.

**THEOREM 2.6** For  $m = 1, 2, \dots$

$$\begin{aligned}
 2.6 \quad \phi^{*m} \circ \exp^{\wedge}(\alpha, \beta, \gamma) &= (P_m(\gamma))^{-1} F \left[ (P_m(\gamma))^{-1} \left( \sum_{l=0}^{m-1} P_l(\gamma) \right) \right. \\
 & \qquad \qquad \qquad \left. (\alpha^2 + \beta^2) + m\gamma^2 \right]
 \end{aligned}$$



COROLLARY 2.7 For  $m = 1, 2, \dots$

$$2.7 \quad \phi^{*m} \circ \exp(x, y, z) = \int_{R^3} E \left[ -(P_m(c))^{\frac{1}{2}} x a, -(P_m(c))^{\frac{1}{2}} y b, -z c \right] \\ F \left[ \sum_{l=0}^{m-1} P_l(c) (a^2 + b^2) + m c^2 \right] da db dc$$

and in particular,

$$2.8 \quad \phi^{*m} \circ \exp(0, 0, 0) = \int_{R^3} F \left[ \sum_{l=0}^{m-1} P_l(c) (a^2 + b^2) + m c^2 \right] da db dc.$$

PROOF

We apply the Fourier inversion theorem to the Schwartz function  $\phi^{*m} \circ \exp^{\wedge}$  to conclude from theorem 2.6 that

$$\phi^{*m} \circ \exp(x, y, z) = \int_{R^3} (P_m(\gamma))^{-1} E(-x\alpha, -y\beta, -z\gamma) \\ F \left[ (P_m(\gamma))^{-1} \sum_{l=0}^{m-1} P_l(\gamma) (\alpha^2 + \beta^2) + m\gamma^2 \right] \\ da db d\gamma .$$

Making the substitutions  $a = (P_m(\gamma))^{-\frac{1}{2}} \alpha$ ,  $b = (P_m(\gamma))^{-\frac{1}{2}} \beta$ ,  $c = \gamma$  we obtain 2.7.

As an immediate consequence of 2.7 we have

COROLLARY 2.8 For  $m=1, 2, \dots$

$$|| \phi^{*m} \circ \exp ||_{\infty} = \phi^{*m} \circ \exp(0, 0, 0) .$$

## Chapter III

### Some Polynomials

In the last section we saw that the behavior of the function  $\phi^{*m} \circ \exp$  is intimately related to the behavior of certain polynomials in one variable. In this section we study the polynomials  $P_m$  of the previous section and also other polynomials derived from them in some detail. The notation of the previous section remains in force.

PROPOSITION 3.1 For  $m=1,2,\dots$

- 1)  $P_m \in R[\gamma^2/4]$  and  $P_m(0)=1$ .
- 2) Letting  $x=\gamma^2/4$  and viewing  $P_m$  as a polynomial in  $x$ ,  $\text{degree } P_m = [m/2]$ . Moreover, if
$$P_m(x) = 1 + a_1x + \dots + a_{[m/2]}x^{[m/2]}$$
then  $a_i$  is a positive integer for  $i = 1, \dots, [m/2]$ . If  $m$  is even, then  $a_{m/2}=1$ .

PROOF

- 1) is a restatement of corollary 2.3.

To prove 2) we proceed by induction on  $m$ , the cases  $m=0,1$  being trivial. Suppose inductively that  $m \geq 1$  and that the proposition has been proved for  $0, \dots, m$ . We consider two cases.

Case 1.  $m$  odd.

We let  $x=\gamma^2/4$  and replace  $\gamma^2/4$  by  $x$  everywhere that  $\gamma^2/4$  appears in the expressions for the polynomials  $P_i(\gamma)$ .

Then by proposition 2.2 and a slight abuse of notation

$$3.1 \quad P_{m+1}(x) = P_m(x) + x \sum_{i=0}^{m-1} P_i(x) .$$

By the inductive hypothesis,

$$\deg P_m(x) = \frac{m-1}{2} = \deg P_{m-1}(x)$$

and for  $i \leq m-2$

$$\deg P_i(x) \leq \frac{m-3}{2}$$

with each of the polynomials  $P_i(x)$ ,  $i=0, \dots, m$  having positive integer coefficients. We thus see that there are positive integers  $a_1, \dots, a_{\frac{m-1}{2}}, b_1, \dots, b_{\frac{m-3}{2}}$  so that

$$P_m(x) + x \sum_{i=0}^{m-2} P_i(x) = 1 + a_1 x + \dots + a_{\frac{m-1}{2}} x^{\frac{m-1}{2}}$$

and

$$xP_{m-1}(x) = x + b_1 x^2 + \dots + b_{\frac{m-3}{2}} x^{\frac{m-1}{2}} + x^{\frac{m+1}{2}}$$

where we have used the inductive hypothesis to conclude that the leading coefficient of  $P_{m-1}(x)$  is 1. From 3.1 and the above expressions we see that  $P_{m+1}(x)$  is a polynomial of degree  $\frac{m+1}{2} = \left\lceil \frac{m+1}{2} \right\rceil$  with positive integer coefficients and leading coefficient 1.

Case 2.  $m$  even.

Proceeding as in case 1 we have that

$$P_{m+1}(x) = P_m(x) + x \sum_{i=0}^{m-1} P_i(x)$$

where  $\deg P_m(x) = \frac{m}{2} = \left[ \frac{m+1}{2} \right]$  and  $\deg P_i(x) \leq \frac{m-2}{2}$  for  $i=0, \dots, m-1$ . The proposition in this case now follows immediately.

This completes the proof of the proposition.

DEFINITION 3.2 For  $m, k=0, 1, \dots$  define  $a_{m,k}$  by

$$\begin{aligned} 3.2 \quad P_m(\gamma) = & a_{m,0} + a_{m,1}(\gamma^2/4) + \dots \\ & + a_{m, [m/2]}(\gamma^2/4)^{[m/2]} \end{aligned}$$

and set  $a_{m,k}=0$  for  $k > [m/2]$ .

Our immediate goal is to obtain a formula for  $a_{m,k}$  in terms of  $m$  and  $k$ .

LEMMA 3.3 For  $m=0, 1, \dots$  and  $k=1, 2, \dots$

$$3.3 \quad a_{m,k} = \sum_{i=0}^{m-2} (m-i-1)a_{i,k-1}$$

PROOF

Since

$$P_m(\gamma) = P_{m-1}(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-2} P_i(\gamma)$$

for  $m \geq 1$  and  $P_0(\gamma)=1$ , we see that

$$a_{m,k} = a_{m-1,k} + \sum_{i=0}^{m-2} a_{i,k-1}.$$

We fix  $k \geq 1$  and proceed by induction on  $m$ . For  $m=0$  the lemma is trivial. Assuming inductively that 3.3 holds for  $m$ ,

$$\begin{aligned}
a_{m+1,k} &= a_{m,k} + \sum_{i=0}^{m-1} a_{i,k-1} \\
&= \sum_{i=0}^{m-2} (m-i-1)a_{i,k-1} + \sum_{i=0}^{m-1} a_{i,k-1} \\
&= a_{m-1,k-1} + \sum_{i=0}^{m-2} (m-i)a_{i,k-1} \\
&= \sum_{i=0}^{m-1} (m-i)a_{i,k-1}
\end{aligned}$$

This completes the proof of the lemma.

At this point we recall the classical fact (see e.g. [6]) that for a non-negative integer  $p$

$$\begin{aligned}
3.4 \quad 1^p + 2^p + \dots + k^p &= \frac{k^{p+1}}{p+1} + \frac{k^p}{2} \\
&\quad + \sum_{n=2}^{p+1} \frac{B_n}{n} \binom{p}{n-1} k^{p-n+1}
\end{aligned}$$

where the numbers  $B_n$  are the Bernoulli numbers which occur as the coefficients of the power series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n z^n, \quad |z| < 2\pi.$$

3.4 is actually a stronger statement than we need. We shall use only the fact that the left side of 3.4 is given by a polynomial in  $k$  of degree  $p+1$ .

LEMMA 3.4 For  $m, k=0, 1, \dots$ ,  $a_{m,k} = S_k(m)$  where  $S_k$  is a polynomial of degree  $2k$ .

PROOF

We note that  $a_{m,0} = 0$  for all  $m$  and proceed by induction

on  $k$ . Assume inductively that the lemma is true for  $k$ .

Using 3.3,

$$\begin{aligned} a_{m,k+1} &= \sum_{i=0}^{m-2} (m-i-1)a_{i,k} \\ &= (m-1) \sum_{i=0}^{m-2} S_k(i) - \sum_{i=0}^{m-2} iS_k(i) \end{aligned}$$

where  $S_k$  is a polynomial of degree  $2k$ . Applying 3.4 to the summation of the various powers of  $i$  occurring in the polynomials  $S_k(i)$  and  $iS_k(i)$  we see that  $a_{m,k+1}$  is given by the difference of two polynomials in  $m$  of degree  $2k+2$ . Thus there is a polynomial  $S_{k+1}$  of degree at most  $2k+2$  such that

$$a_{m,k+1} = S_{k+1}(m) .$$

Since  $a_{0,k+1} = \dots = a_{2k+1,k+1} = 0$  we see that  $S_{k+1}$  must in fact be of degree  $2k+2$ . This completes the inductive step and thereby the proof of the lemma.

We are now in a position to determine an explicit formula for  $a_{m,k}$ .

PROPOSITION 3.5 For  $m, k=0, 1, \dots$

$$3.5 \quad a_{m,k} = \frac{1}{(2k)!} \prod_{i=0}^{2k-1} (m-i)$$

PROOF

By lemma 3.4, for  $k \geq 1$ ,

$$a_{m,k} = S_k(m)$$

where  $S_k$  is a polynomial of degree  $2k$  and

$$S_k(0) = \dots = S_k(2k-1) = 0 .$$

Thus for  $k \geq 1$ ,

$$a_{m,k} = c \cdot m(m-1) \dots (m-(2k-1))$$

for some constant  $c$ . By proposition 3.1

$$1 = a_{2k,k} = c \cdot 2k(2k-1) \dots (1) = c(2k)! .$$

Thus  $c=1/(2k)!$  . This proves the proposition for  $k \geq 1$ . For  $k=0$ ,  $S_k \equiv 1$  and the proposition is trivially true.

If we consider the integral of 2.7 which yields  $\phi^{*m} \circ \exp(x,y,z)$  we see that we need to determine not only the polynomials  $P_1(\gamma)$  but also certain sums of these polynomials.

DEFINITION 3.6 For  $m=0,1,\dots$  define polynomials  $\Gamma_m$  by

$$3.6 \quad \sum_{l=0}^{m-1} P_l(\gamma) = m + \Gamma_m(\gamma)$$

We note that  $\Gamma_m(0)=0$  and that if  $\Gamma_m$  is viewed as an element in  $R[\gamma^2/4]$  then  $\deg \Gamma_m = [(m-1)/2]$  for  $m \geq 1$ , and  $\Gamma_0=0$ .

DEFINITION 3.7 For  $m,k=0,1,\dots$  define  $\tau_{m,k}$  by

$$3.7 \quad \Gamma_m(\gamma) = \tau_{m,0} + \tau_{m,1} \left(\frac{\gamma^2}{4}\right) + \dots + \tau_{m, \left[\frac{m-1}{2}\right]} \left(\frac{\gamma^2}{4}\right)^{\left[\frac{m-1}{2}\right]}$$

and set  $\tau_{m,k}=0$  for  $k > [(m-1)/2]$ . Note that  $\tau_{m,0}=0$  for all  $m$ .

PROPOSITION 3.8 For  $m=0,1,\dots$  and  $k=1,2,\dots$

$$3.8 \quad \tau_{m,k} = \frac{1}{(2k+1)!} \prod_{i=0}^{2k} (m-i)$$

PROOF

From 3.6 and 3.7 we see that for  $k \geq 1$ ,

$$\tau_{m,k} = \sum_{i=0}^{m-1} a_{i,k} , m=0,1,\dots$$

Thus  $\tau_{m,k}=0$  for  $m=0,\dots,2k$ . Since by proposition 3.4  $a_{i,k}$  is given by a polynomial in  $i$  of degree  $2k$  we have by equation 3.4 that  $\tau_{m,k}$  is given by a polynomial in  $m$  of degree  $2k+1$ . Thus

$$\tau_{m,k} = c \cdot m(m-1)\dots(m-2k)$$

for some constant  $c$ . Since  $\tau_{2k+1,k}=a_{2k,k}=1$  we have that

$$1 = c \cdot (2k+1)(2k)\dots(1) = c \cdot (2k+1)! .$$

Thus  $c=1/(2k+1)!$  and the proof of the proposition is complete.

COROLLARY 3.9 For  $c \neq 0$

$$3.9 \quad \lim_{m \rightarrow \infty} \frac{1}{m} \Gamma_m(m^{-\frac{1}{2}}c) = \infty .$$

PROOF

Applying 3.7 and 3.8 we have

$$\begin{aligned} \frac{1}{m} \Gamma_m(m^{-\frac{1}{2}}c) &= \frac{1}{m} \sum_{k=1}^{[(m-1)/2]} \frac{m(m-1)\dots(m-2k)}{(2k+1)!} \cdot \frac{c^{2k}}{m^{k+1}4^k} \\ &= \sum_{k=1}^{[(m-1)/2]} \frac{m(m-1)\dots(m-2k)}{(2k+1)! m^{k+1}} \left(\frac{c}{2}\right)^{2k} . \end{aligned}$$

For fixed  $c$  and  $k$  each of the terms in the above sum tend to  $\infty$  individually as  $m \rightarrow \infty$ . This proves the corollary.



## Chapter IV

### Asymptotic Behavior of the Random Walk

The notation of the previous sections remains in force.  
In corollary 2.7 we saw that

$$\phi^{*m} \circ \exp(x, y, z) = \int_{R^3} E \left[ -(P_m(c))^{\frac{1}{2}} x a, -(P_m(c))^{\frac{1}{2}} y b, -z c \right] \\ F \left[ \sum_{l=0}^{m-1} P_l(c) (a^2 + b^2) + m c^2 \right] da db dc .$$

Alternately,

$$\phi^{*m} \circ \exp(x, y, z) = \int_{R^3} E \left[ -(P_m(c))^{\frac{1}{2}} x a, -(P_m(c))^{\frac{1}{2}} y b, -z c \right] \\ F \left[ (m + \Gamma_m(c)) (a^2 + b^2) + m c^2 \right] da db dc .$$

Making a change of variables this last integral becomes

$$m^{-3/2} \int_{R^3} E \left[ -(P_m(m^{-\frac{1}{2}}c))^{\frac{1}{2}} m^{-\frac{1}{2}} x a, -(P_m(m^{-\frac{1}{2}}c))^{\frac{1}{2}} m^{-\frac{1}{2}} y b, -m^{-\frac{1}{2}} z c \right] \\ F \left[ m^{-1} (m + \Gamma_m(m^{-\frac{1}{2}}c)) (a^2 + b^2) + c^2 \right] da db dc$$

Again making a change of variables we have that

$$\phi^{*m} \circ \exp(x, y, z) = m^{-3/2} \int_{R^3} E \left[ - \left( \frac{P_m(m^{-\frac{1}{2}}c)}{m + \Gamma_m(m^{-\frac{1}{2}}c)} \right)^{\frac{1}{2}} x a, \right. \\ \left. - \left( \frac{P_m(m^{-\frac{1}{2}}c)}{m + \Gamma_m(m^{-\frac{1}{2}}c)} \right)^{\frac{1}{2}} y b, -m^{-\frac{1}{2}} z c \right] \\ \frac{m}{m + \Gamma_m(m^{-\frac{1}{2}}c)} G(a, b, c) da db dc .$$

We integrate with respect to  $da db$  to conclude that

$$4.1 \quad \phi^{*m} \circ \exp(x, y, z) = m^{-3/2} \int_{\mathbb{R}} \frac{m}{m + \Gamma_m(m^{-1/2}c)} E(-m^{-1/2}zc) \\ F\left[\frac{P_m(m^{-1/2}c)}{m + \Gamma_m(m^{-1/2}c)} (x^2 + y^2)\right] \\ G(c) \quad dc$$

and in particular,

$$4.2 \quad \phi^{*m} \circ \exp(0, 0, 0) = m^{-3/2} \int_{\mathbb{R}} \frac{m}{m + \Gamma_m(m^{-1/2}c)} G(c) \, dc$$

In the next theorem we utilize 4.1 and 4.2 to show that, as in the abelian case, the rate at which the Gaussian random walk on  $N_3$  "wanders to  $\infty$ " is determined by the behavior of  $\phi^{*m} \circ \exp(0, 0, 0)$  as  $m \rightarrow \infty$ .

**THEOREM 4.1** For  $k > 0$  let  $B_k = \{(x, y, z) : (x^2 + y^2 + z^2)^{1/2} \leq k\}$ .

Then for  $(x, y, z) \in B_k$ ,

$$\phi^{*m} \circ \exp(0, 0, 0) \geq \phi^{*m} \circ \exp(x, y, z) \geq G(k) \phi^{*m} \circ \exp(0, 0, 0) .$$

Hence,

$$\int_{B_k} \phi^{*m} \circ \exp(x, y, z) \, dx dy dz = O(\phi^{*m} \circ \exp(0, 0, 0))$$

and

$$\phi^{*m} \circ \exp(0, 0, 0) = O\left(\int_{B_k} \phi^{*m} \circ \exp(x, y, z) \, dx dy dz\right)$$

**PROOF**

The first inequality was established in corollary 2.8. To prove the second equality we consider the expressions in

4.1 and 4.2 and make some rather crude estimates. Now, applying 2.2 and 3.6

$$\begin{aligned} P_m(m^{-\frac{1}{2}}c) &= P_{m-1}(m^{-\frac{1}{2}}c) + \frac{c^2}{4m} \sum_{l=0}^{m-2} P_l(m^{-\frac{1}{2}}c) \\ &= P_{m-1}(m^{-\frac{1}{2}}c) + \frac{c^2}{4m} (\Gamma_{m-1}(m^{-\frac{1}{2}}c) + m-1) . \end{aligned}$$

Thus,

$$\begin{aligned} \frac{P_m(m^{-\frac{1}{2}}c)}{m + \Gamma_m(m^{-\frac{1}{2}}c)} &= \frac{P_{m-1}(m^{-\frac{1}{2}}c)}{m + \Gamma_m(m^{-\frac{1}{2}}c)} + \frac{c^2}{4m} \frac{m-1 + \Gamma_{m-1}(m^{-\frac{1}{2}}c)}{m + \Gamma_m(m^{-\frac{1}{2}}c)} \\ &< 1 + \frac{c^2}{4m} . \end{aligned}$$

Hence, for  $(x,y,z) \in B_k$

$$\begin{aligned} \phi^{*m} \circ \exp(x,y,z) &> m^{-3/2} \int_{\mathbb{R}} \frac{m}{m + \Gamma_m(m^{-\frac{1}{2}}c)} E(-m^{-\frac{1}{2}}zc) G(c) \\ &\quad F\left[\left(1 + \frac{c^2}{4m}\right)(x^2 + y^2)\right] dc \\ &\geq m^{-3/2} F(x^2 + y^2) \int_{\mathbb{R}} \frac{m}{m + \Gamma_m(m^{-\frac{1}{2}}c)} G(c) dc \\ &\geq G(k) \phi^{*m} \circ \exp(0,0,0) . \end{aligned}$$

In the above calculation we implicitly used the fact that since all of the real-valued functions, when viewed as functions of  $c$ , are even, only  $\text{Re}(E(-m^{-\frac{1}{2}}zc))$  affects the integral. This completes the proof of the theorem.

DEFINITION 4.2 For  $m=1,2,\dots$  set

$$\begin{aligned} I_m &= (\phi \circ \exp^{*m}(0,0,0))^{-1} \phi^{*m} \circ \exp(0,0,0) \\ &= m^{3/2} \phi^{*m} \circ \exp(0,0,0) \\ &= \int_{\mathbb{R}} \frac{1}{1 + m^{-1} \Gamma_m(m^{-1/2}c)} G(c) \, dc . \end{aligned}$$

From the integral defining  $I_m$  we see immediately that  $I_m < 1$ . Thus we see that not only is the non-abelian walk transient but also that it "wanders to  $\infty$ " at least as rapidly as does the abelian walk. In fact the non-abelian walk "wanders to  $\infty$ " more rapidly than does the abelian walk as one sees by applying corollary 3.9 and the dominated convergence theorem to conclude that  $I_m \rightarrow 0$  as  $m \rightarrow \infty$ . It is now natural to inquire about the rate at which  $I_m$  tends to 0. This question will be answered rather precisely in theorem 4.4.

LEMMA 4.3 For  $a > 0$ ,  $c \in \mathbb{R}$

$$4.3 \quad \lim_{m \rightarrow \infty} \sum_{k=[am]+1}^{\infty} (mc)^k \frac{1}{(2k+1)!} = 0,$$

the limit being uniform on compact subsets of  $\mathbb{R}$ .

PROOF

$$\begin{aligned} \sum_{k=[am]+1}^{\infty} |(mc)^k \frac{1}{(2k+1)!}| &\leq \sum_{k=[am]+1}^{\infty} \frac{|mc|^k}{(am)^k} \frac{1}{(k+1)!} \\ &= \sum_{k=[am]+1}^{\infty} (|c|/a)^k \frac{1}{(k+1)!} , \end{aligned}$$

the latter series tending uniformly to 0 on compacta as  $m \rightarrow \infty$ .

## THEOREM 4.4

- a)  $\lim_{m \rightarrow \infty} m^p I_m = 0$  for  $p < \frac{1}{2}$
- b)  $0 < \liminf_{m \rightarrow \infty} m^{\frac{1}{2}} I_m \leq \limsup_{m \rightarrow \infty} m^{\frac{1}{2}} I_m < \infty$
- c)  $\lim_{m \rightarrow \infty} m^p I_m = \infty$  for  $p > \frac{1}{2}$ .

## PROOF

Although a) and c) follow immediately from b) we shall prove them independently since little extra work is involved in doing so and the proofs are instructive.

Since  $\Gamma_m$  and  $G$  are even functions,

$$m^p I_m = 2 \int_0^{\infty} \frac{m^p}{1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}} c)} G(c) dc.$$

Since  $\Gamma_m$  is a monotone increasing function we have that

$$\begin{aligned} 4.4 \quad m^p I_m &> \int_0^{m^{-\frac{1}{2}}} \frac{m^p}{1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}} c)} G(c) dc \\ &> G(1) m^{p-\frac{1}{2}} (1 + m^{-1} \Gamma_m(m^{-1}))^{-1}. \end{aligned}$$

Now, combining 3.7 with 3.8 and setting  $\gamma = m^{-1}$  we have

$$\begin{aligned} m^{-1} \Gamma_m(m^{-1}) &= \sum_{k=1}^{[(m-1)/2]} \frac{(m-1) \dots (m-2k)}{(2k+1)!} \frac{1}{4^k} \frac{1}{m^{2k}} \\ &< \sum_{k=1}^{[(m-1)/2]} \frac{1}{(2k+1)!} \frac{1}{4^k}. \end{aligned}$$

This shows that  $m^{-1} \Gamma_m(m^{-1})$  remains bounded as  $m \rightarrow \infty$ . Applying

this result to 4.4 we see that

$$4.5 \quad \liminf_{m \rightarrow \infty} m^{\frac{1}{2}} I_m > 0$$

and

$$4.6 \quad \lim_{m \rightarrow \infty} m^p I_m = \infty \quad \text{for } p > \frac{1}{2}.$$

On the other hand,

$$\int_0^{m^{-\frac{1}{2}}} (1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}} c))^{-1} m^p G(c) \, dc < m^{p-\frac{1}{2}}$$

and thus

$$4.7 \quad \lim_{m \rightarrow \infty} \int_0^{m^{-\frac{1}{2}}} (1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}} c))^{-1} m^p G(c) \, dc = 0 \quad \text{for } p < \frac{1}{2}$$

and

$$4.8 \quad \limsup_{m \rightarrow \infty} \int_0^{m^{-\frac{1}{2}}} (1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}} c))^{-1} m^{\frac{1}{2}} G(c) \, dc \leq 1.$$

To complete the proof of the theorem we must consider the behavior of

$$\int_{m^{-\frac{1}{2}}}^{\infty} (1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}} c))^{-1} G(c) \, dc.$$

Consider the map

$$\begin{aligned} c &\mapsto 1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}} c) \\ &= 1 + \sum_{k=1}^{[(m-1)/2]} \frac{(m-1) \dots (m-2k)}{(2k+1)!} \left( \frac{c^2}{4m} \right)^k. \end{aligned}$$

For  $m \geq 2$

$$\begin{aligned}
 4.9 \quad 1 + m^{-1} \Gamma(m^{-\frac{1}{2}}c) &> 1 + \sum_{k=1}^{\lfloor m/4 \rfloor} \frac{(m-1) \dots (m-2k)}{(2k+1)!} \left(\frac{c^2}{4m}\right)^k \\
 &> 1 + \sum_{k=1}^{\lfloor m/4 \rfloor} \frac{(\frac{1}{2}m)^{2k}}{(2k+1)!} \left(\frac{c^2}{4m}\right)^k \\
 &= \sum_{k=0}^{\lfloor m/4 \rfloor} \frac{1}{(2k+1)!} \left(\frac{mc^2}{16}\right)^k
 \end{aligned}$$

from whence we see that for all  $p$  and all  $c \neq 0$

$$\lim_{m \rightarrow \infty} m^{-p} \cdot m^{-1} \Gamma_m(m^{-\frac{1}{2}}c) = \infty,$$

the limit being uniform away from 0. Thus for all  $p$ ,

$$4.10 \quad \lim_{m \rightarrow \infty} \int_1^\infty m^p (1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}}c))^{-1} G(c) dc = 0.$$

We next consider the behavior of the integral

$$\int_{m^{-\frac{1}{2}}}^1 m^p (1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}}c))^{-1} G(c) dc.$$

Define

$$4.11 \quad f(r) = \sum_{k=0}^{\infty} \frac{r^k}{(2k+1)!}, \quad r \in \mathbb{R}.$$

By 4.9 and lemma 4.3 we conclude that for  $K_0$  fixed,  $0 < K_0 < 1$ , and for  $m$  sufficiently large

$$4.12 \quad 1 + m^{-1} \Gamma_m(m^{-\frac{1}{2}}c) > K_0 f(mc^2/16), \quad c \in [0, 1].$$

Now, for  $r > 0$

$$4.13 \quad f(r) = r^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!}.$$

We also have that

$$\begin{aligned}
4.14 \quad e^{\sqrt{r}} &= \sum_{k=0}^{\infty} \frac{\sqrt{r}^k}{k!} \\
&= 1 + \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+2}}{(2k+2)!} + \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!} \\
&= 1 + \sqrt{r} \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+2)!} + \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!} .
\end{aligned}$$

Combining 4.13 and 4.14 we conclude that for  $r > 0$

$$\begin{aligned}
\sqrt{r}f(r) &= \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!} \\
&> \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+2)!} \\
&= r^{-\frac{1}{2}}(e^{\sqrt{r}} - 1 - \sqrt{r}f(r))
\end{aligned}$$

from whence

$$f(r) > (r + \sqrt{r})^{-1}(e^{\sqrt{r}} - 1) .$$

Thus for  $c \in [m^{-\frac{1}{2}}, 1]$

$$4.15 \quad f\left(\frac{mc^2}{16}\right) \geq \left(\frac{mc^2}{16} + \frac{\sqrt{mc}}{4}\right)^{-1} e^{\sqrt{mc}/4} (1 - e^{-\frac{1}{4}}) .$$

Setting  $K = K_0(1 - e^{-\frac{1}{4}})$  we have by 4.12 and 4.15 that

$$4.16 \quad 1 + m^{-1}\Gamma_m(m^{-\frac{1}{2}}c) > K\left(\frac{mc^2}{16} + \frac{\sqrt{mc}}{4}\right)^{-1} e^{\sqrt{mc}/4}$$

for  $c \in [m^{-\frac{1}{2}}, 1]$  and  $m$  sufficiently large. Thus for large  $m$



$$\begin{aligned}
& \int_{m^{-1/2}}^1 (1 + m^{-1} \Gamma_m(m^{-1/2}c))^{-1} G(c) \, dc \\
& < K^{-1} \int_{m^{-1/2}}^1 \left( \frac{mc^2}{16} + \frac{\sqrt{mc}}{4} \right) e^{-\sqrt{mc}/4} \, dc \\
& = K^{-1} \int_{1/4}^{\sqrt{m}/4} (x^2 + x) e^{-x \cdot 4m^{-1/2}} \, dx \\
& < K^{-1} 4m^{-1/2} \int_0^{\infty} (x^2 + x) e^{-x} \, dx .
\end{aligned}$$

Since the last integral above is finite we conclude that

$$4.17 \quad \limsup_{m \rightarrow \infty} \sqrt{m} \int_{m^{-1/2}}^1 (1 + m^{-1} \Gamma_m(m^{-1/2}c))^{-1} G(c) \, dc < \infty$$

and for  $p < 1/2$

$$4.18 \quad \lim_{m \rightarrow \infty} m^p \int_{m^{-1/2}}^1 (1 + m^{-1} \Gamma_m(m^{-1/2}c))^{-1} G(c) \, dc = 0.$$

Finally, 4.6 proves c), 4.7, 4.10 and 4.18 together prove a), and 4.5, 4.8, 4.10 and 4.17 together prove b).

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## Vita

David Robert Fischer, son of Robert and Eleanor Fischer, was born in Belleville, Illinois on February 12, 1951. He received his early schooling in New Jersey and Louisiana. In 1968 he entered Louisiana State University as an Alumni Federation Scholar. He received the B.S. degree, magna cum laude, in electrical engineering in 1972. In 1972 he entered the graduate program in electrical engineering. During this period he was supported by a National Science Foundation Traineeship and contributed several joint publications to the engineering literature. He received the M.S.E.E. degree in 1974.

He entered the graduate program in the Department of Mathematics in the fall of 1974, where he was employed as a teaching assistant. The academic year 1976-77 was spent as a teaching assistant at the University of Connecticut where he conducted dissertation research under the direction of his advisor, Professor Leonard F. Richardson, who was on leave at that institution. While studying for his doctoral degree he attended several mathematical symposia and presented his results before a national meeting of the American Mathematical Society.

In September, 1978, Mr. Fischer joined the faculty of the Mathematics Department of the University of Utah.

Mr. Fischer is a member of Phi Eta Sigma, Eta Kappa Nu,

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**EXAMINATION AND THESIS REPORT**

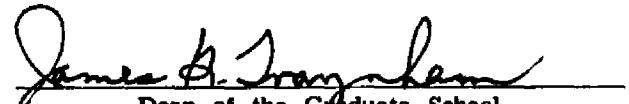
**Candidate:** David Robert Fischer

**Major Field:** Mathematics--Analysis

**Title of Thesis:** The Gaussian Random Walk on the Heisenberg Group

**Approved:**

  
Major Professor and Chairman

  
Dean of the Graduate School

**EXAMINING COMMITTEE:**









**Date of Examination:**

June 8, 1978